Chhattisgarh Institute of Technology, Jashpur

Department of Science & Humanities

Subject- Applied Mathematics

Subject Code-2000B04AT014 (NEP)

Semester- Second

Session- 2025-26

Prepared By-

Dr. Arun Kumar, Lecturer Mathematics

Lecture Notes on Integral Calculus, Applications of Integral Calculus, First-Order First-Degree Differential Equations, Numerical Solutions of Equations, and Numerical Integration

Overview

These lecture notes present five linked topics from engineering mathematics: integral calculus, applications of integration, first-order first-degree differential equations, numerical solutions of equations, and numerical integration.[cite:6] [cite:7] The notes are written as a classroom-style resource with definitions, standard formulas, worked explanation, and one illustrative example in each major topic.[cite:6] [cite:7]

Integral calculus studies accumulation and antiderivatives, while applications of integration connect the definite integral to area, volume, work, mass, hydrostatic force, arc length, surface area, and center of mass.[cite:6] First-order differential equations model changing systems such as mixing, population growth, and falling bodies, and several standard forms can be solved systematically.[cite:7] Numerical methods extend these ideas when exact algebraic or analytic methods are difficult or impossible to apply directly.[cite:3] [cite:4]

1. Integral Calculus

1.1 Meaning of integration

Integration is the inverse process of differentiation and is used to recover a function from its derivative.[cite:6] When a function represents a rate of change, its integral represents the accumulated quantity over an interval.[cite:6]

There are two central forms of integration:

- **Indefinite integral:**
  $$\int f(x)\,dx = F(x) + C$$
  where $F'(x) = f(x)$.
- **Definite integral:**
  $$\int_a^b f(x)\,dx = F(b) - F(a)$$
  provided F is an antiderivative of f.

The Fundamental Theorem of Calculus links differentiation and integration, showing that accumulation through a definite integral can be evaluated using antiderivatives.[cite:6]

1.2 Basic integration formulas

Some standard formulas used repeatedly are:

- $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C$, for $n \neq -1$
- $\int \dfrac{1}{x} \, dx = \ln|x| + C$
- $\int e^x \, dx = e^x + C$
- $\int a^x \, dx = \dfrac{a^x}{\ln a} + C$, for $a > 0, a \neq 1$
- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \dfrac{1}{1+x^2} \, dx = \tan^{-1} x + C$

These formulas are combined with algebraic simplification, substitution, and integration by parts in practical problem solving.[cite:6]

1.3 Methods of integration

Substitution

Substitution is useful when the integrand contains a function and its derivative in recognizable form. The idea is to put $u = g(x)$, then $du = g'(x)dx$, reducing the integral to a simpler one.

Example form:

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du$$

Integration by parts

Integration by parts comes from the product rule for derivatives and is written as

$$\int u \, dv = uv - \int v \, du$$

It is useful for products like algebraic-trigonometric, algebraic-exponential, or logarithmic functions.

Partial fractions

Partial fraction decomposition is used for rational functions when the denominator factors. The integrand is split into simpler fractions that can be integrated term by term.

1.4 Geometric interpretation of definite integral

For a nonnegative function, the definite integral $\int_a^b f(x)\,dx$ gives the area under the curve between $x = a$ and $x = b$.[cite:6] If a function changes sign, the integral gives net signed area, so regions below the axis contribute negatively.[cite:6]

This idea is the basis for many engineering applications because the same mathematical structure appears whenever a small quantity is summed continuously.[cite:6]

1.5 Worked example: basic definite integral

Evaluate

$$\int_0^2 (3x^2 + 4x + 1)\,dx$$

Solution:

$$\int (3x^2 + 4x + 1)\,dx = x^3 + 2x^2 + x$$

Applying the limits:

$$\int_0^2 (3x^2 + 4x + 1)\,dx = [x^3 + 2x^2 + x]_0^2$$

$$= (8 + 8 + 2) - 0 = 18$$

So the value of the definite integral is 18.

1.6 Important points for revision

- Integration reverses differentiation.
- A definite integral gives total accumulation on an interval.[cite:6]
- The Fundamental Theorem of Calculus connects limits of sums, antiderivatives, and area.[cite:6]
- Algebraic skill and correct choice of method are essential in integration.

2. Applications of Integral Calculus

Applications of integration arise because many physical and geometric quantities can be built by adding infinitely many small contributions.[cite:6] OpenStax's applications chapter includes areas between curves, volumes by slicing, cylindrical shells, arc length, surface area, work, mass, hydrostatic force, and centers of mass.[cite:6]

2.1 Area under a curve

If $f(x) \geq 0$ on $[a, b]$, then the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$A = \int_a^b f(x)\,dx$$

This is the most basic application of the definite integral.[cite:6]

2.2 Area between two curves

If $f(x) \geq g(x)$ on $[a, b]$, then the area between the curves is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

If the curves cross each other, the interval must be split at points of intersection so that top minus bottom is used correctly.[cite:6]

2.3 Volume by slicing

The volume of a solid with cross-sectional area $A(x)$ at position x is

$$V = \int_a^b A(x)\,dx$$

This general principle supports the disk method, washer method, and other slicing methods.[cite:6]

2.4 Volumes of revolution

When a region is revolved about an axis, common formulas are:

- Disk method:

$$V = \pi \int_a^b [R(x)]^2\,dx$$

- Washer method:

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right) dx$$

- Shell method:

$$V = 2\pi \int_a^b (\text{radius})(\text{height})\,dx$$

OpenStax treats slicing, disks, washers, and cylindrical shells as standard approaches for such problems.[cite:6]

2.5 Arc length and surface area

Arc length of $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Surface area of revolution about the x-axis is

$$S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

These formulas are often harder to evaluate than area and volume integrals.[cite:6]

2.6 Physical applications

Definite integrals are used to compute mass when density varies, work done by a variable force, pumping work, and hydrostatic force on submerged surfaces.[cite:6] These applications are important in mechanics, fluid systems, civil engineering, and materials analysis.[cite:6]

For example:

- If linear density is $\rho(x)$, then mass is $m = \int_a^b \rho(x)\,dx$.
- If force is $F(x)$, then work is $W = \int_a^b F(x)\,dx$.
- If pressure varies with depth, hydrostatic force is found by integrating pressure times strip area over depth.[cite:6]

2.7 Moments and center of mass

The center of mass is the balancing point of a body or lamina, and moments help locate this point.[cite:6] This concept is important in structural analysis and mechanics because it describes how mass is distributed.[cite:6]

For a one-dimensional rod with density $\rho(x)$,

$$\bar{x} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$$

2.8 Worked example: area between curves

Find the area enclosed between $y = 2x + 3$ and $y = x^2$ between their points of intersection.

Step 1: Find the points of intersection.

$$2x + 3 = x^2$$

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

So the curves meet at $x = -1$ and $x = 3$.

Step 2: Determine which curve is above the other on the interval.
At $x = 0$, the line gives 3 and the parabola gives 0, so $2x + 3$ is above x^2.

Step 3: Form the integral.

$$A = \int_{-1}^{3} [(2x + 3) - x^2] dx$$

Step 4: Integrate.

$$A = \left[x^2 + 3x - \frac{x^3}{3} \right]_{-1}^{3}$$

At $x = 3$:

$$9 + 9 - 9 = 9$$

At $x = -1$:

$$1 - 3 + \frac{1}{3} = -\frac{5}{3}$$

Hence,

$$A = 9 - \left(-\frac{5}{3} \right) = \frac{32}{3}$$

So the required area is $\frac{32}{3}$ square units.

2.9 Revision points

- Area, volume, mass, work, and force can all be expressed as definite integrals.[cite:6]
- Correct limits and correct geometric interpretation are crucial.
- In area between curves, use upper function minus lower function.[cite:6]
- In physical problems, identify the small element first, then integrate.

3. Differential Equations of First Order and First Degree

A first-order differential equation involves the first derivative only, and first degree means the derivative appears to the first power.[cite:7] The general first-order equation may be written as $dy/dt = f(y, t)$, but there is no single formula that solves all such equations.[cite:7]

Paul's Online Math Notes lists several important types: linear equations, separable equations, exact equations, Bernoulli equations, substitution-based equations, intervals of validity, modeling problems, equilibrium solutions, and Euler's method.[cite:7]

3.1 Basic definitions

A differential equation is an equation involving derivatives of an unknown function. The order is determined by the highest derivative present, so first-order equations contain only the first derivative.[cite:7]

A solution of a differential equation is a function that satisfies the equation when substituted into it. An initial condition, such as $y(x_0) = y_0$, selects one particular solution from a family.

3.2 Variable separable equations

A separable equation can be written in the form

$$\frac{dy}{dx} = g(x)h(y)$$

or equivalently

$$\frac{dy}{h(y)} = g(x)dx$$

Then both sides can be integrated.

General procedure:

1. Separate variables.
2. Integrate both sides.
3. Add constant of integration.
4. Use the initial condition if given.

3.3 Linear first-order equations

A linear first-order equation has the standard form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

The integrating factor is

$$I.F. = e^{\int P(x)dx}$$

After multiplying the equation by the integrating factor, the left-hand side becomes the derivative of a product, which can then be integrated.[cite:7]

3.4 Exact differential equations

An equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then there exists a function $\phi(x,y)$ such that

$$\frac{\partial \phi}{\partial x} = M, \qquad \frac{\partial \phi}{\partial y} = N$$

and the solution is $\phi(x,y) = C$.[cite:7]

3.5 Bernoulli equation

A Bernoulli equation has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

with $n \neq 0, 1$. A substitution such as $v = y^{1-n}$ converts it into a linear equation.[cite:7]

3.6 Modeling interpretation

First-order equations model systems where the present rate of change depends on current state and input.[cite:7] Common examples include population growth, cooling, mixing of solutions, and motion with resistance.[cite:7]

3.7 Worked example: separable equation

Solve

$$\frac{dy}{dx} = 3xy$$

subject to $y = 2$ when $x = 0$.

Step 1: Separate variables.

$$\frac{dy}{y} = 3x\,dx$$

Step 2: Integrate.

$$\int \frac{1}{y} dy = \int 3x \, dx$$

$$\ln |y| = \frac{3x^2}{2} + C$$

Step 3: Exponentiate.

$$y = Ce^{\frac{3x^2}{2}}$$

(Here the constant is renamed.)

Step 4: Apply the initial condition.
At $x = 0, y = 2$, so

$$2 = Ce^0 = C$$

Hence,

$$y = 2e^{\frac{3x^2}{2}}$$

This is the required solution.

3.8 Euler’s method as a first-order approximation tool

Euler’s method is a numerical procedure for approximating solutions of a first-order differential equation when an exact form is difficult to obtain.[cite:7] Starting from (x_0, y_0), the recursion is

$$y_{n+1} = y_n + hf(x_n, y_n)$$

where h is the step size.[cite:7]

3.9 Revision points

- First-order first-degree equations involve only the first derivative and only to the first power.[cite:7]
- Common solvable forms are separable, linear, exact, and Bernoulli.[cite:7]
- The integrating factor is central to linear equations.[cite:7]
- Initial conditions convert the general solution into a particular solution.

4. Numerical Solutions of Equations

Numerical root-finding methods are used when an equation cannot be solved exactly by algebraic methods or when an approximate solution is sufficient for practical work.[cite:3] Such methods are central in engineering computation because many nonlinear equations arise from design formulas, physical models, and optimization conditions.[cite:3]

4.1 Meaning of root of an equation

A root of the equation $f(x) = 0$ is a value of x for which the function becomes zero. Numerical methods aim to compute approximate root values with desired accuracy.[cite:3]

4.2 Why numerical methods are needed

Many equations such as transcendental equations, high-degree polynomials, and model-generated nonlinear equations cannot be solved by simple formulas.[cite:3] In such cases, iterative methods generate a sequence of approximations converging to the root.[cite:3]

4.3 Bisection method

The bisection method requires a continuous function on an interval $[a, b]$ where $f(a)$ and $f(b)$ have opposite signs. This guarantees a root in the interval by the Intermediate Value Theorem.[cite:3]

Algorithm:

1. Compute midpoint $c = \frac{a+b}{2}$.
2. Evaluate $f(c)$.
3. Replace the subinterval that still has sign change.
4. Repeat until the interval is sufficiently small.

Advantages:

- Simple and reliable.
- Guaranteed convergence if assumptions are satisfied.

Disadvantage:

- Converges slowly.

4.4 Newton-Raphson method

Newton's method uses tangent lines and has iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

It often converges much faster than bisection when the initial guess is good and $f'(x_n) \neq 0$. [cite:3]

Advantages:

- Fast convergence near the root.

Disadvantages:

- Requires derivative.

- May diverge if the starting value is poor.

4.5 Fixed-point iteration

If $f(x) = 0$ can be rewritten as $x = g(x)$, then iteration may be formed as

$$x_{n+1} = g(x_n)$$

Convergence depends on the choice of $g(x)$ and the behavior of its derivative near the fixed point.[cite:3]

4.6 Error ideas

Approximation quality is measured by absolute error, relative error, or the difference between successive iterates. In practical computation, stopping criteria often use a tolerance such as

$$|x_{n+1} - x_n| < \varepsilon$$

where ε is a prescribed small positive number.

4.7 Worked example: bisection method

Find an approximate root of

$$x^3 - x - 2 = 0$$

using the bisection method.

Let

$$f(x) = x^3 - x - 2$$

Check values:

$$f(1) = 1 - 1 - 2 = -2$$

$$f(2) = 8 - 2 - 2 = 4$$

Since the sign changes between 1 and 2, a root lies in $[1, 2]$.

Iteration 1:

$$c_1 = \frac{1 + 2}{2} = 1.5$$

$$f(1.5) = 3.375 - 1.5 - 2 = -0.125$$

The sign change is between 1.5 and 2.

Iteration 2:

$$c_2 = \frac{1.5 + 2}{2} = 1.75$$

$$f(1.75) = 5.359375 - 1.75 - 2 = 1.609375$$

The sign change is between 1.5 and 1.75.

Iteration 3:

$$c_3 = \frac{1.5 + 1.75}{2} = 1.625$$

$$f(1.625) = 4.291015625 - 1.625 - 2 = 0.666015625$$

The sign change is between 1.5 and 1.625.

Iteration 4:

$$c_4 = \frac{1.5 + 1.625}{2} = 1.5625$$

$$f(1.5625) = 3.814697265625 - 1.5625 - 2 = 0.252197265625$$

The sign change is between 1.5 and 1.5625.

Iteration 5:

$$c_5 = \frac{1.5 + 1.5625}{2} = 1.53125$$

$$f(1.53125) = 3.590850830078125 - 1.53125 - 2 = 0.059600830078125$$

The sign change is between 1.5 and 1.53125.

Iteration 6:

$$c_6 = \frac{1.5 + 1.53125}{2} = 1.515625$$

$$f(1.515625) = 3.4826927185058594 - 1.515625 - 2 = -0.0329322814941406$$

So the root lies between 1.515625 and 1.53125, giving approximate root

$$x \approx 1.52$$

4.8 Practical remarks

- Bisection is preferred when guaranteed convergence is important.[cite:3]
- Newton-Raphson is preferred when speed is important and a good initial estimate is available.[cite:3]
- Choice of stopping criterion affects accuracy and computational cost.

5. Numerical Integration

Numerical integration is used when an antiderivative is hard to find or when the function is known only through tabulated or measured values.[cite:4] This is common in laboratory work, engineering computation, and simulation where data may be discrete rather than symbolic.[cite:4]

5.1 Basic idea

A definite integral

$$I = \int_a^b f(x)\,dx$$

is approximated by replacing the curve with simple geometric shapes or interpolation formulas over subintervals.[cite:4]

5.2 Trapezoidal rule

If the interval $[a, b]$ is divided into n equal parts of width

$$h = \frac{b-a}{n}$$

and ordinates are $y_0, y_1, \ldots, y_n$, then the composite trapezoidal rule is

$$\int_a^b f(x)dx \approx \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})]$$

This rule approximates the curve by straight-line segments joining consecutive points.[cite:4]

5.3 Simpson's one-third rule

When the number of subintervals n is even, Simpson's one-third rule is

$$\int_a^b f(x)dx \approx \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

This method is generally more accurate than the trapezoidal rule for smooth functions because it uses parabolic approximation over pairs of intervals.

5.4 Simpson's three-eighth rule

When the number of subintervals is a multiple of 3, Simpson's three-eighth rule may be used:

$$\int_a^b f(x)dx \approx \frac{3h}{8}[y_0 + y_n + 3(\text{sum of interior terms not multiples of 3}) + 2(\text{sum of in}$$

5.5 Error understanding

The error in numerical integration depends on interval width, smoothness of the function, and the formula used. In general, smaller step sizes improve accuracy but require more computation.

5.6 Euler's method connection

Paul's notes place Euler's method in the first-order differential equations chapter as an approximation technique for differential equations.[cite:7] Although it is not a direct numerical integration formula for definite integrals, it is conceptually related because it builds an approximate accumulated solution step by step.[cite:7]

5.7 Worked example: trapezoidal rule

Approximate

$$\int_0^4 x^2\,dx$$

using the trapezoidal rule with $n = 4$.

Step 1: Compute step size.

$$h = \frac{4-0}{4} = 1$$

Step 2: Tabulate values.

x	0	1	2	3	4
$y = x^2$	0	1	4	9	16

Step 3: Apply formula.

$$\int_0^4 x^2 dx \approx \frac{1}{2}[0 + 16 + 2(1 + 4 + 9)]$$

$$= \frac{1}{2}[16 + 28] = 22$$

So the trapezoidal approximation is 22.

For comparison, the exact value is

$$\int_0^4 x^2 dx = \left[\frac{x^3}{3}\right]_0^4 = \frac{64}{3} \approx 21.33$$

Thus the trapezoidal result is close but not exact.

5.8 Worked example: Simpson's one-third rule

Approximate the same integral using Simpson's one-third rule with $n = 4$.

Using the same values,

$$\int_0^4 x^2 dx \approx \frac{1}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1}{3}[0 + 16 + 4(1 + 9) + 2(4)]$$

$$= \frac{1}{3}[16 + 40 + 8] = \frac{64}{3}$$

So Simpson's one-third rule gives the exact value here because the function is a polynomial of degree 2.

5.9 Revision points

- Numerical integration is useful for difficult integrals and tabulated data.[cite:4]
- Trapezoidal rule uses line segments; Simpson's rules use polynomial approximation.
- Smaller subintervals usually improve accuracy.
- Always check whether the condition on number of subintervals is satisfied before applying Simpson's rules.

6. Formula Sheet

Integral calculus formulas

- $\int f'(x)dx = f(x) + C$
- $\int_a^b f(x)dx = F(b) - F(a)$
- $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C, n \neq -1$
- $\int \dfrac{1}{x} dx = \ln|x| + C$
- $\int e^x dx = e^x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \cos x dx = \sin x + C$
- $\int u\, dv = uv - \int v\, du$

Application formulas

- Area under curve: $A = \int_a^b f(x)dx$
- Area between curves: $A = \int_a^b [f(x) - g(x)]dx$
- Volume by slicing: $V = \int_a^b A(x)dx$
- Disk method: $V = \pi \int_a^b R^2 dx$
- Washer method: $V = \pi \int_a^b (R^2 - r^2)dx$
- Shell method: $V = 2\pi \int_a^b (\text{radius})(\text{height})dx$
- Arc length: $L = \int_a^b \sqrt{1 + (y')^2}dx$
- Work: $W = \int_a^b F(x)dx$

Differential equations formulas

- Separable form: $\dfrac{dy}{dx} = g(x)h(y)$
- Linear form: $\dfrac{dy}{dx} + P(x)y = Q(x)$
- Integrating factor: $e^{\int P(x)dx}$
- Exactness condition: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$
- Bernoulli form: $\dfrac{dy}{dx} + P(x)y = Q(x)y^n$
- Euler's method: $y_{n+1} = y_n + hf(x_n, y_n)$

Numerical methods formulas

- Bisection midpoint: $c = \dfrac{a+b}{2}$
- Newton-Raphson: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
- Fixed point: $x_{n+1} = g(x_n)$
- Trapezoidal rule: $\dfrac{h}{2}[y_0 + y_n + 2\sum y_i]$
- Simpson's one-third rule: $\dfrac{h}{3}[y_0 + y_n + 4\sum y_{odd} + 2\sum y_{even}]$

7. Typical Examination Questions

1. Evaluate a definite integral using the Fundamental Theorem of Calculus.
2. Find the area bounded by two given curves.[cite:6]
3. Find the volume generated by revolving a region about an axis.[cite:6]
4. Solve a separable differential equation with an initial condition.[cite:7]
5. Solve a linear first-order differential equation using integrating factor.[cite:7]
6. Test whether a differential equation is exact and solve it if exact.[cite:7]
7. Find a root of a nonlinear equation by bisection or Newton-Raphson method.[cite:3]
8. Approximate a definite integral by trapezoidal or Simpson's rule.[cite:4]

8. Study Guidance

A good sequence for study is to begin with antiderivatives and definite integrals, then move to geometric and physical applications, and only after that study first-order differential equations.[cite:6] [cite:7] Numerical methods should be learned alongside exact methods so that the limitations of symbolic techniques become clear in practical problems.[cite:3] [cite:4]

For revision, it is effective to memorize standard formulas, practice one solved example of each type, and then attempt mixed problems where the appropriate method is not stated in advance. Accuracy in algebra, signs, limits, and interpretation matters as much as memorizing formulas.

9. Closing Note

These topics form a connected unit in applied mathematics: integration measures accumulation, applications convert integrals into geometry and physics, differential equations model change, and numerical methods provide approximations when exact methods become difficult.[cite:6] [cite:7] [cite:3] [cite:4] A student who understands both the formulas and the meaning of the small element in each method will be able to solve most standard university-level problems in this syllabus.